STRANGE PRODUCTS OF PROJECTIONS

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Abstract. Let H be an infinite dimensional Hilbert space. We show that there exist three orthogonal projections X_1, X_2, X_3 onto closed subspaces of H such that for every $0 \neq z_0 \in H$ there exist $k_1, k_2, \dots \in \{1,2,3\}$ so that the sequence of iterates defined by $z_n = X_{k_n} z_{n-1}$ does not converge in norm.

Introduction

Let K be a fixed natural number and let $X_1, X_2, \dots, X_K$ be a family of K closed subspaces of a Hilbert space H. Let $z_0 \in H$ and $k_1, k_2, \dots \in \{1, 2, \dots, K\}$ be arbitrary. Consider the sequence of vectors $\{z_n\}_{n=1}^{\infty}$ defined by

$$z_n = X_{k_n} z_{n-1}, \tag{1}$$

where X_k denotes also the orthogonal projection of H onto the subspace X_k. The sequence $\{z_n\}$ converges weakly by a theorem of Amemiya and Ando [AA]. If each projection appears in the sequence $\{P_{k_n}\}$ infinitely many times, then this limit is equal to the orthogonal projection of z_0 onto $\bigcap_{i=1}^{K} X_i$.

If $K = 2$ then the sequence $\{z_n\}$ converges even in norm according to a classical result of von Neumann [N]. An elementary geometric proof of this theorem can be found in [KR].

If $K \geq 3$, then additional assumptions are needed to ensure the norm-convergence. That $\{z_n\}$ converges if H is finite dimensional was originally proved by Práger [Pr]; this also follows, of course, from [AA].

If H is infinite dimensional, but the sequence $\{k_n\}$ is periodic, the sequence $\{z_n\}$ converges in norm according to Halperin [Ha]. The result was generalized to quasiperiodic sequences by Sakai [S]. Recall that the sequence $\{k_n\}$ is quasiperiodic if there exists $r \in \mathbb{N}$ such that $\{k_m, k_{m+1}, \dots, k_{m+r}\} = \{1, 2, \dots, K\}$ for each $m \in \mathbb{N}$.

The question of norm-convergence if H infinite dimensional, $K \geq 3$ and $\{k_n\}$ arbitrary, posed in [AA], was open for a long time. In 2012

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Paszkiewicz [P] constructed an ingenious example of 5 subspaces of an infinite dimensional Hilbert space and of a sequence $\{z_n\}$ of the form (1) which does not converge in norm. An important input towards the construction originates in Hundal's example ([H], see also [K] and [MR]) of two closed convex subsets of an infinite dimensional Hilbert space and a sequence of alternating projections onto them which does not converge in norm.

E. Kopecká and V. Müller resolved in [KM] fully the question of Amemia and Ando. They refined Paszkiewicz's construction to get a rather complicated example of three subspaces of an infinite dimensional Hilbert space and of a sequence $\{z_n\}$ of the form (1) which does not converge in norm.

In Section 2 of this paper we considerably simplify the construction of [KM], resulting in Theorem 2.6. Moreover, we strengthen the statement on existence of one badly behaved point to the statement that *all points* (except for zero) in a Hilbert space H can be badly behaved. Namely, in Theorem 3.5 we show that in every infinite dimensional Hilbert space H, there exist three orthogonal projections X_1, X_2, X_3 onto closed subspaces of H such that for every $0 \neq z_0 \in H$ there exist $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $z_n = X_{k_n} z_{n-1}$ does not converge in norm.

If we allow five projections instead of just three, only two sequences of indices are needed to get divergence at every point. In Theorem 3.3 we show that there exists a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ with the following property. For every infinite dimensional Hilbert space H there are closed subspaces X, Y, X_1, X_2, X_3 of H so that if $0 \neq z \in H$, and $u_0 = Xz$ and $v_0 = XYz$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^{\infty}$ or $\{v_n\}_{n=1}^{\infty}$ defined by $u_n = X_{k_n} u_{n-1}$ and $v_n = X_{k_n} v_{n-1}$ does not converge in norm.

The paper is organized as follows.

In Section 1 we observe that if we have control over the number of appearances of a projection in a product, then replacing the projection by another one, close in the norm to the original one, does not change the product much.

Let u and v be two orthonormal vectors. In Section 2 we construct two almost identical subspaces X and Y and a product of projections $\psi(u \vee v, X, Y)$ onto the span $u \vee v$ of u and v, and the spaces X and Y so that $\psi(u \vee v, X, Y)u \approx v$. Using the results of Section 1, we "glue" countably many copies of such triples of spaces together. We thus obtain three projections and a divergent sequence $\{z_n\}$ of iterates (1) which trail very closely along the quartercircles connecting consecutive vectors of an orthonormal sequence.

In Section 3 we deduce from the existence of three subspaces and one point the iterates of which do not converge the existence of other three subspaces such that for *any nonzero point* in the space certain iterates of the projections on these three subspaces do not converge.

Notation. In the entire paper H is a Hilbert space; we will explicitly mention when we need H to be infinite dimensional. Let $B(H)$ be the space of bounded linear operators from H to H.

For a closed subspace X of H we denote by X also the orthogonal projection onto X. For $M, N \subset H$ we denote by $\bigvee M$ the closed linear hull of M, and by $M \vee N$ the closed linear hull of $M \cup N$. Similarly, we use $\vee x$ and $x \vee y$ for $x, y \in H$.

For $m \in \mathbb{N}$ let $\mathcal{S}_m$ be the free semigroup with generators $a_1, \dots, a_m$. If $\varphi = a_{i_r} \cdots a_{i_1} \in \mathcal{S}_m$ (for some $r \in \mathbb{N}$ and $i_j \in \{1, \dots, m\}$) and $A_1, \dots, A_m \in B(H)$, then we write $\varphi(A_1, \dots, A_m) = A_{i_r} \cdots A_{i_1} \in B(H)$. If $X_1, \dots, X_m$ are closed subspaces of H, then $\varphi(X_1, \dots, X_m) = X_{i_r} \cdots X_{i_1} \in B(H)$ is the respective product of orthogonal projections.

Denote by $|\varphi| = r$ the "length" of the word φ, and by $|\varphi_i|$ the number of "occurrences" of a_i in the word φ. Then $\sum_{i=1}^{m} |\varphi_i| = |\varphi| = r$.

1. Continuous dependence of words on the letters

Orthogonal projections are 1-Lipschitz mappings and so are their products. This allows us to construct our examples "imprecisely". If for certain z_0 the sequence of iterates $\{z_n\}$ defined by (1) does not converge in norm, the same is true of such sequences of iterates starting from a point w from a small enough neighborhood of z.

Now assume, we have control on the number of appearances of a letter a in the word φ. If we plug in projections, substituting A for a, or instead of A a projection B close enough in norm to A, the resulting product does not change much.

Lemma 1.1. *Let $\psi \in \mathcal{S}_n$ for some $n \in \mathbb{N}$. Assume $A_i, B_i, E \in B(H)$, $i \in \{1, 2, \dots, n\}$ are contractions so that each A_i commutes with E. Then*

$$\|\psi(A_1, \dots, A_n)E - \psi(B_1, \dots, B_n)E\| \leq \sum_{1 \leq i \leq n} |\psi_i| \, \|A_i E - B_i E\|.$$

Proof. We proceed by induction on $r = |\psi|$. For $r = 0$ the statement is obvious as $\psi(A_1, \dots, A_n) = \psi(B_1, \dots, B_n)$ is the identity mapping on H. Let the statement be valid upto some r and let $|\psi| = r + 1$. Then

$\psi = \varphi a_j$ for some $\varphi \in \mathcal{S}_n$ with $|\varphi| = r$ and $j \in \{1, \dots, n\}$. Hence

$$\begin{aligned}
&\|\psi(A_1, \dots, A_n)E - \psi(B_1, \dots, B_n)E\| \\
=&\|\varphi(A_1, \dots, A_n)A_jE - \varphi(B_1, \dots, B_n)B_jE\| \\
\le&\|\varphi(A_1, \dots, A_n)A_jE - \varphi(B_1, \dots, B_n)A_jE\| \\
&+ \|\varphi(B_1, \dots, B_n)A_jE - \varphi(B_1, \dots, B_n)B_jE\| \\
\le&\|\varphi(A_1, \dots, A_n)E - \varphi(B_1, \dots, B_n)E\| + \|\varphi(B_1, \dots, B_n)\| \cdot \|A_jE - B_jE\| \\
\le&\|A_jE - B_jE\| + \sum_{1\le i\le n} |\varphi_i| \; \|A_iE - B_iE\| \\
\le& \sum_{1\le i\le n} |\psi_i| \; \|A_iE - B_iE\|,
\end{aligned}$$

by induction, and since $A_jE = EA_j$, $\|A_j\| \le 1$ and $\|\varphi(B_1, \dots, B_n)\| \le 1$. □

The following two lemmata are straightforward colloraries of the above. We include them for easy reference.

Lemma 1.2. *Let $\psi \in \mathcal{S}_3$. Suppose E, W, X, X', Y, Y', Z are subspaces of H so that $W, X, Y \subset E$ and $X', Y' \perp E$. Then*

$$\|\psi(W, X, Y)E - \psi(Z, X \vee X', Y \vee Y')E\| \le |\psi_1| \cdot \|ZE - W\|.$$

Proof. This is a corollary of Lemma 1.1 for $n = 3$, since $WE = W$, and $X = XE = (X \vee X')E$, and similarly, $Y = YE = (Y \vee Y')E$. □

The next statement is a corollary of Lemma 1.1 for $E = H$.

Lemma 1.3. *Let $\varphi \in \mathcal{S}_n$ and $A_1, \dots A_n$, $B_1, \dots, B_n$ be projections for some $n \in \mathbb{N}$. Then*

$$\|\varphi(A_1, \dots, A_n) - \varphi(B_1, \dots, B_n)\| \le |\varphi| \cdot \max_{i=1,\dots,n} \|A_i - B_i\|.$$

2. Projectional iterates of a point may diverge

Let H be an infinite dimensional Hilbert space. According to [KM], there exist three closed subspaces $X_1, X_2, X_3 \subset H$, such that the sequence of iterates $\{w_n\}_{n=1}^{\infty}$ defined by $w_n = X_{k_n} w_{n-1}$ does not converge in norm. In this section we simplify the proof of this statement.

The example is “glued” together from finite dimensional blocks. In each of these blocks three subspaces and a finite product of projections are constructed so that the product maps a given normalized vector u with an arbitrary precision on a normalized vector v orthogonal to u.

For $\varepsilon > 0$ let $k = k(\varepsilon)$ be the smallest positive integer k such that $(\cos \frac{\pi}{2k})^k > 1 - \varepsilon$. That is, if u and v are two orthonormal vectors, and

we project u consecutively onto the lines $g_1, \dots, g_k$ dividing the right angle between u and v into k angles of size $\frac{\pi}{2k}$, then we land at v with error at most ε

Projecting onto a line g_j can be arbitrarily well approximated by iterating projections between two subspaces intersecting at this particular line. In the next lemma we call these spaces $W = u \vee v$ and X_j'. By introducing a small error, we modify X_j' into X_j, to ensure that $X_1 \subset \cdots \subset X_k$. This will enable us later to replace the projections on each of the spaces X_j by the product $(XYX)^{s(j)}$ of projections onto the largest space $X = X_k$ and its suitable small variation Y. Alltogether, instead of projecting onto several spaces to get from u to v, we project only on three of them: W, X, and Y. We get a suitable product ψ of these projections so that $\psi(W, X, Y)u \approx v$.

The next lemma is modified from [P], the proof is taken from [KM].

Lemma 2.1. *Let $\varepsilon > 0$. Then there exists $\varphi \in \mathcal{S}_{k(\varepsilon)+1}$ with the following properties:*

Suppose X is a subspace of H so that $\dim X = \infty$, and $u, v \in X$ are so that $\|u\| = \|v\| = 1$ and $u \perp v$. Then there exist subspaces $X_1 \subset \cdots \subset X_{k(\varepsilon)} \subset X$ so that $\dim X_j = j + 1$ for all $j \in \{1, \dots, k(\varepsilon)\}$, and

$$\left\|\varphi(W, X_1, \dots, X_{k(\varepsilon)})u - v\right\| < 2\varepsilon,$$

where $W = u \vee v$.

Proof. Write $k := k(\varepsilon)$. Choose orthonormal vectors $z_0, z_1, \dots, z_{k-1} \in W^\perp \cap X$.

Let $\xi = \frac{\pi}{2k}$. For $j = 0, \dots, k$, let $h_j = u \cos j\xi + v \sin j\xi$ be the points on the quarter circle connecting $h_0 = u$ to $h_k = v$. We construct inductively a rapidly decreasing sequence of nonnegative numbers $\alpha_0 > \alpha_1 > \cdots > \alpha_{k-1} > \alpha_k = 0$ in the following way. Choose $\alpha_0 \in (0, 1)$ arbitrarily. Let $1 \le j \le k - 1$ and suppose that $\alpha_0, \dots, \alpha_{j-1}$ and subspaces $X_1 \subset \cdots \subset X_{j-1}$ have already been constructed. Set

$$X_j' = \vee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \dots, h_{j-1} + \alpha_{j-1} z_{j-1}, h_j\}.$$

Since $W \cap X_j' = \vee h_j$, we have $(X_j' W X_j')^r x \to (\vee h_j)x$ for each $x \in H$ as $r \to \infty$, by [N]. As both spaces are finite dimensional, there exists $r(j) \in \mathbb{N}$ such that

$$\left\|(X_j' W X_j')^{r(j)} - (\vee h_j)\right\| < \frac{\varepsilon}{k}.$$

Let $\alpha_j > 0$ be so small that

$$\left\|(X_j W X_j)^{r(j)} - (\vee h_j)\right\| < \frac{\varepsilon}{k}, \tag{2}$$

where

$$X_j = \vee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \dots, h_{j-1} + \alpha_{j-1} z_{j-1}, h_j + \alpha_j z_j\}.$$

Suppose that $X_1 \subset X_2 \subset \cdots \subset X_{k-1}$ have already been constructed. Set formally $\alpha_k = 0$ and $X_k = X_k' = \vee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \dots, h_{k-1} + \alpha_{k-1} z_{k-1}, h_k\}$. Find $r(k) \in \mathbb{N}$ such that (2) is true also for $j = k$. Then $v = h_k \in X_k$. Let $\varphi \in \mathcal{S}_{k+1}$ and $\psi \in \mathcal{S}_k$ be defined by

$$\varphi(c, b_1, \dots, b_k) = (b_k c b_k)^{r(k)} \cdots (b_1 c b_1)^{r(1)},$$

and $\psi(a_1, \dots, a_k) = a_1 \dots a_k$. Then

$$\begin{aligned}&\big\|\varphi(W, X_1, \dots, X_k) - \vee h_k \cdots \vee h_1\big\| \\ =&\big\|\psi((X_k W X_k)^{r(k)}, \dots, (X_1 W X_1)^{r(1)}) - \psi(\vee h_k, \dots, \vee h_1)\big\| < k \cdot \frac{\varepsilon}{k},\end{aligned}$$

by Lemma 1.3. Hence

$$\begin{aligned}&\big\|\varphi(W, X_1, \dots, X_k)u - v\big\| \\ \leq&\big\|\varphi(W, X_1, \dots, X_k)u - (\vee h_k \cdots \vee h_1)u\big\| + \big\|(\vee h_k \cdots \vee h_1)u - v\big\| < 2\varepsilon.\end{aligned}$$

It follows from the construction, that the resulting word $\varphi \in \mathcal{S}_{k(\varepsilon)+1}$ does not depend on the particular X, u, and v.

□

Projections onto an increasing family of n finite dimensional spaces can be replaced by projections onto just two spaces: onto the largest space in the family and onto a suitable small variation of it. The next lemma is generalized from [P], its proof from [KM].

Lemma 2.2. *Let $k \in \mathbb{N}$ and $\varepsilon > 0$, $\eta > 0$, and $a > 0$ be given. There exist natural numbers $a < s(k) < s(k-1) < \cdots < s(1)$ with the following property. Suppose $X_1 \subset X_2 \subset \cdots \subset X_k \subset X \subset E \subset H$ are closed subspaces so that X is separable and* $\dim X^\perp \cap E = \infty$. *Then there exists a closed subspace $Y \subset E$ such that $X \cap Y = \{0\}$, $\|X - Y\| < \eta$ and for each $j \in \{1, \dots, k\}$,*

$$\big\|(XYX)^{s(j)} - X_j\big\| < \varepsilon.$$

Proof. We can assume $0 < \eta < 1$. First we fix $0 < \beta_{k+1} < \eta/2$ and choose $s(k) > a$ such that $1/(1 + \beta_{k+1}^2)^{s(k)} < \varepsilon$. Next we inductively choose numbers

$$\beta_k, s(k-1), \beta_{k-1}, s(k-2), \dots, s(1), \beta_1$$

such that

$$\beta_{k+1} > \beta_k > \cdots > \beta_1 > 0,$$
$$a < s(k) < s(k-1) < \cdots < s(1),$$
$$\frac{1}{(1+\beta_{j+1}^2)^{s(j)}} < \varepsilon \text{ and } \left|\frac{1}{(1+\beta_j^2)^{s(j)}} - 1\right| < \varepsilon$$

for $j = k, \dots, 1$. We show that these s_j's are as required.

Let $\{e_i\}_{i \in I}$ be an at most countable orthonormal basis in X such that there are index sets $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_k \subset I_{k+1} = I$ with the property, that $\{e_i\}_{i \in I_j}$ is an orthonormal basis in X_j for $j \in \{1, \dots, k\}$. For $i \in I_j \setminus I_{j-1}$ define $\gamma_i = \beta_j$. Let $\{w_i\}_{i \in I}$ be an orthonormal system in $X^\perp \cap E$. We construct Y as the closed linear span of the vectors $e_i + \gamma_i w_i$, $i \in I$. Note that if Y is constructed in this way, we have for $m \in \mathbb{N}$ and $i \in I$,

$$(XYX)^m e_i = \frac{e_i}{(1+\gamma_i^2)^m}. \tag{3}$$

If $x = \sum_{i \in I} a_i e_i \in X$, then by (3),

$$\begin{aligned} \left\|(XYX)^{s(j)}x - X_j x\right\|^2 &= \Big\|\sum_{i \in I} a_i \frac{e_i}{(1+\gamma_i^2)^{s(j)}} - \sum_{i \in I_j} a_i e_i\Big\|^2 \\ &= \sum_{i \in I_j} a_i^2 \left(1 - \frac{1}{(1+\gamma_i^2)^{s(j)}}\right)^2 + \sum_{i \in I \setminus I_j} a_i^2 \frac{1}{(1+\gamma_i^2)^{2s(j)}} \qquad (4) \\ &\le \varepsilon^2 \sum_{i \in I} a_i^2 = \varepsilon^2 \|x\|^2. \end{aligned}$$

For any $z \in H$ we have

$$(XYX)^{s(j)}z - X_j z = (XYX)^{s(j)}(Xz) - X_j(Xz),$$

since $X_j \subset X$. Hence by (4) for $j \in \{1, \dots, k\}$,

$$\left\|(XYX)^{s(j)} - X_j\right\| < \varepsilon.$$

It is easy to see that $\|X-Y\|<\eta$. Indeed, for any $0\neq z\in H$,

$$\begin{aligned}\|Xz-Yz\|^2 =&\|\sum_{i\in I}\langle e_i,z\rangle e_i-\sum_{i\in I}\frac{1}{1+\gamma_i^2}\langle e_i+\gamma_i w_i,z\rangle(e_i+\gamma_i w_i)\|^2\\ \leq&\sum_{i\in I}\frac{1}{(1+\gamma_i^2)^2}[\gamma_i^2\langle e_i,z\rangle-\gamma_i\langle w_i,z\rangle]^2\\ &+\frac{\eta^2}{4}\sum_{i\in I}[\frac{1}{1+\gamma_i^2}\langle e_i+\gamma_i w_i,z\rangle]^2\\ \leq&2(\eta/2)^4\|z\|^2+2(\eta/2)^2\|z\|^2+(\eta^2/4)\|z\|^2\\ <&\eta^2\|z\|^2\end{aligned}$$

We have used that $0<\gamma_i<\beta_{k+1}<\eta/2<1$, and in the third line the estimate $(a-b)^2\leq 2a^2+2b^2$. Since $0<\gamma_i$ for all $i\in I$, we have that $X\cap Y=\{0\}$. □

Again, let u and v be orthornormal, $W=u\vee v$, and $\varepsilon>0$. In the next lemma we construct a word ψ and a two "almost parallel" spaces X and Y so that $\big\|\psi(W,X,Y)u-v\big\|<2\varepsilon$. Importantly, the number of appearances of W in ψ depends *only* on ε. On the contrary, the closer together the spaces X and Y are, the larger is their number of appearances in ψ.

Lemma 2.3. *For every $\varepsilon>0$, there exists $N=N(\varepsilon)$, so that for every $\eta>0$, there exists $\psi\in\mathcal{S}_3$ so that $|\psi_1|\leq N$ with the following property.*

Given subspaces $X\subset E\subset H$ so that X is separable and $\dim X=\dim X^\perp\cap E=\infty$, and $u,v\in X$ are so that $\|u\|=\|v\|=1$ and $u\perp v$, there exists a subspace $Y\subset E$ such that $X\cap Y=\{0\}$, $\|X-Y\|<\eta$, and

$$\big\|\psi(W,X,Y)u-v\big\|<3\varepsilon,$$

where $W=u\vee v$.

Proof. Let $\varepsilon>0$ be given. Let $\varphi\in\mathcal{S}_{k(\varepsilon)+1}$ be as in Lemma 2.1, and $N=|\varphi_1|$. Let $\eta>0$ be given. For $k=k(\varepsilon)$, ε replaced by $\varepsilon/|\varphi|$, the given η, and $a=1$ choose the natural numbers $s(k)<s(k-1)<\cdots<s(1)$ according to Lemma 2.2. To define the word ψ, replace for each $i\in\{2,\ldots,k+1\}$ the letter a_i in φ by $(a_2a_3a_2)^{s(i-1)}$. With a slight abuse of notation, but certainly more understandably,

$$\psi(W,X,Y)=\varphi(W,(XYX)^{s(1)},\ldots,(XYX)^{s(k)}).$$

Clearly, $|\psi_1|=|\varphi_1|=N$.

Let $u, v \in X \subset E$ as in the lemma be given. According to Lemma 2.1, there exist subspaces $X_1 \subset \cdots \subset X_k \subset X$ so that

$$\left\|\varphi(W, X_1, \ldots, X_k)u - v\right\| < 2\varepsilon,$$

where $W = u \vee v$. By Lemma 2.2 there exists a subspace $Y \subset E$ such that $X \cap Y = \{0\}$, $\|X - Y\| < \eta$ and for each $j \in \{1, \ldots, k\}$,

$$\left\|(XYX)^{s(j)} - X_j\right\| < \varepsilon/|\varphi|.$$

Then

$$\begin{aligned}
&\left\|\psi(W, X, Y)u - v\right\| \\
&\quad= \|\varphi(W, (XYX)^{s(1)}, \ldots, (XYX)^{s(k)})u - v\| \\
&\quad\le \left\|\varphi(W, (XYX)^{s(1)}, \ldots, (XYX)^{s(k)})u - \varphi(W, X_1, \ldots, X_k)u\right\| \\
&\qquad + \left\|\varphi(W, X_1, \ldots, X_k)u - v\right\| < 3\varepsilon
\end{aligned}$$

by Lemma 1.3. □

Next we show that we have relative freedom of choice for the three spaces verifying $\psi(W, X, Y)u \approx v$.

Lemma 2.4. *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$, so that for every $\eta > 0$, there exists $\psi \in \mathcal{S}_3$ with the following property.*

Given subspaces $X \subset E \subset H$ so that X is separable and $\dim X = \dim X^\perp \cap E = \infty$, and $u, v \in X$ are so that $\|u\| = \|v\| = 1$ and $u \perp v$, there exists a subspace $Y \subset E$ such that $X \cap Y = \{0\}$, $\|X - Y\| < \eta$ with the following property. If $W = u \vee v$ and X', Y', Z are subspaces such that $X', Y' \subset E^\perp$ and $\|W - ZE\| < \delta$ then

$$\left\|\psi(Z, X \vee X', Y \vee Y')u - v\right\| < 4\varepsilon.$$

Proof. Given an $\varepsilon > 0$, choose $N \in \mathbb{N}$ according to Lemma 2.3 and put $\delta = \varepsilon/N$. For these ε and N and a given η choose ψ according to Lemma 2.3, and for a given subspace X choose also Y according to this lemma. Let X', Y', Z be as above. Lemma 1.2 implies, that

$$\begin{aligned}
&\left\|\psi(Z, X \vee X', Y \vee Y')u - v\right\| \\
&\quad\le \left\|\psi(Z, X \vee X', Y \vee Y')u - \psi(W, X, Y)u\right\| + \left\|\psi(W, X, Y)u - v\right\| \\
&\quad\le |\psi_1|\|W - ZE\| + 3\varepsilon \le N\varepsilon/N + 3\varepsilon = 4\varepsilon.
\end{aligned}$$

□

Now we glue the finite dimensional steps together. Given an orthonormal sequence $\{e_i\}_{i=1}^\infty$ with an infinite dimensional orthocomplement, we construct three spaces X, Y, Z and words $\Psi_i \in \mathcal{S}_3$ so that $\Psi_i(Z, X, Y)e_i \approx e_{i+1}$ for all $i \in \mathbb{N}$.

Lemma 2.5. *For any $\varepsilon_i > 0$, $i = 1, 2, \dots$, there exist $\Psi_i \in \mathcal{S}_3$ with the following property.*

Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in H with an infinite dimensional orthogonal complement. Then there are three closed subspaces $X, Y, Z \subset H$ so that

$$\left\|\Psi_i(Z, X, Y)e_i - e_{i+1}\right\| < 4\varepsilon_i.$$

Proof. Use Lemma 2.4 to define $\delta_i = \delta(\varepsilon_i)$. Put $\eta_i = \min\{\delta_{i-1}, \delta_{i+1}\}/2$, where $\delta_0 = 1$. Again, use Lemma 2.4 to choose $\psi_i \in \mathcal{S}_3$. For even $i \in \mathbb{N}$ put $\Psi_i = \psi_i$, for the odd i define $\Psi_i(Z, X, Y) = \psi_i(Y, X, Z)$.

Let E_i, $i \in \mathbb{N}$ be closed infinite dimensional subspaces of H so that

$$\begin{aligned} &e_i, e_{i+1} \in E_i, \\ &\vee e_{i+1} = E_i E_{i+1} = E_{i+1} E_i, \text{ and} \\ &E_i \perp E_j \text{ if } |i - j| \geq 2. \end{aligned}$$

Fix also closed subspaces $X_i \subset E_i$, so that $e_i, e_{i+1} \in X_i$, and $\dim X_i = \dim(X_i^{\perp} \cap E_i) = \infty$.

By Lemma 2.4, there exist closed subspaces $Y_i \subset E_i$ so that

$$\left\|\psi_i(Z_i, X_i \vee X', Y_i \vee Y')e_i - e_{i+1}\right\| < 4\varepsilon, \tag{5}$$

whenever $W_i = e_i \vee e_{i+1}$ and X', Y', Z_i are subspaces such that $X', Y' \subset E_i^{\perp}$ and $\|W_i - Z_i E_i\| < \delta_i$. Put $Y_0 = \vee e_1$ and

$$X = \bigvee_{i=1}^{\infty} X_i,\ Y = \bigvee_{k=0}^{\infty} Y_{2k},\ Z = \bigvee_{k=0}^{\infty} Y_{2k+1}.$$

Then for any $i \in \mathbb{N}$ we have $X = X_i \vee X_i'$ for a suitable $X_i' \perp E_i$. Further we distinguish between two cases: i is even and i is odd.

If $i = 2k$ is even, then also $Y = Y_i \vee Y_i'$ for a suitable $Y_i' \perp E_i$. Since $ZE_i = (Y_{i-1} \vee Y_{i+1})E_i$, we have

$$\begin{aligned} \|W_i - ZE_i\| &= \|(X_{i-1} \vee X_{i+1})E_i - (Y_{i-1} \vee Y_{i+1})E_i\| \\ &= \|(X_{i-1} + X_{i+1})E_i - (Y_{i-1} + Y_{i+1})E_i\| \\ &= \|(X_{i-1} - Y_{i-1})E_i + (X_{i+1} - Y_{i+1})E_i\| \\ &\leq \|X_{i-1} - Y_{i-1}\| + \|X_{i+1} - Y_{i+1}\| \leq \eta_{i-1} + \eta_{i+1} \\ &\leq \min\{\delta_{i-2} + \delta_i\}/2 + \min\{\delta_i + \delta_{i+2}\}/2 \leq \delta_i. \end{aligned}$$

Hence by (5),

$$\left\|\psi_i(Z, X, Y)e_i - e_{i+1}\right\| < 4\varepsilon_i.$$

If $i = 2(k-1)$ is odd, then $Z = Y_i \vee Y_i'$ for a suitable $Y_i' \perp E_i$, and similarly as above one can show that

$$\|W_i - YE_i\| \leq \delta_i.$$

Hence by (5),

$$\left\|\psi_i(Y,X,Z)e_i - e_{i+1}\right\| < 4\varepsilon_i.$$

□

Finally we prove the main result of this section, Theorem 2.6 of [KM].

Theorem 2.6. *There exists a sequence $k_1, k_2, \dots \in \{1,2,3\}$ with the following property. If H is an infinite dimensional Hilbert space and $0 \neq w_0 \in H$, then there exist three closed subspaces $X_1, X_2, X_3 \subset H$, such that the sequence of iterates $\{w_n\}_{n=1}^\infty$ defined by $w_n = X_{k_n} w_{n-1}$ does not converge in norm. (Here X_n denotes also the orthogonal projection onto X_n.) More precisely, the spaces $X_1, X_2, X_3 \subset H$ intersect only at the origin, and the sequence $\{w_n\}$, although weakly convergent to the origin, stays in norm bounded away from zero.*

Proof. For $\varepsilon_i = 9^{-i}$ choose Ψ_i as in Lemma 2.5. Let $e_1 = w_0/|w_0|$. Fix an orthonormal sequence $\{e_i\}_{i=1}^\infty$ with an infinite dimensional complement. Choose closed subspaces $X, Y, Z \subset H$ according to Lemma 2.5 and call them X_1, X_2, X_3. Write for short $A_i = \Psi_i(Z, X, Y)$. Since $\|A_i\| \le 1$, we have by induction

$$\begin{aligned}
&\|A_n A_{n-1} \dots A_1 e_1 - e_{n+1}\| \\
&\quad \le \|A_n A_{n-1} \dots A_2(A_1 e_1 - e_2)\| + \|A_n A_{n-1} \dots A_2 e_2 - e_{n+1}\| \\
&\quad \le 4\varepsilon_1 + \|A_n A_{n-1} \dots A_3(A_2 e_2 - e_3)\| + \|A_n A_{n-1} \dots A_3 e_3 - e_{n+1}\| \\
&\quad \le 4\varepsilon_1 + 4\varepsilon_2 + \dots + \|A_n e_n - e_{n+1}\| \le 4(9^{-1} + \dots + 9^{-n}) \le 1/2
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{e_i\}$ is an orthonormal sequence, the norm-limit $\lim_{n\to\infty} A_n A_{n-1} \dots A_1 e_1$ does not exist. □

3. All points may have diverging projectional iterates

In this section we strengthen the statement on existence of one badly behaved point to the statement that *all points* (except for zero) in a Hilbert space H can be badly behaved. Namely, in Theorem 3.5 we show that in every infinite dimensional Hilbert space H, there exist three orthogonal projections X_1, X_2, X_3 onto closed subspaces of H such that for every $0 \neq z_0 \in H$ there exist $k_1, k_2, \dots \in \{1,2,3\}$ so that the sequence of iterates defined by $z_n = X_{k_n} z_{n-1}$ does not converge in norm.

If we allow five projections instead of just three, only two sequences of indices are needed to get non-convergence at every point. In Theorem 3.3 we show that there exists a sequence $k_1, k_2, \dots \in \{1,2,3\}$ with the following property. For every infinite dimensional Hilbert space H there are closed subspaces X, Y, X_1, X_2, X_3 of H so that if $0 \neq z \in H$,

and $u_0 = Xz$ and $v_0 = XYz$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^\infty$ or $\{v_n\}_{n=1}^\infty$ defined by $u_n = X_{k_n} u_{n-1}$ and $v_n = X_{k_n} v_{n-1}$ does not converge in norm.

First we prove an auxiliary statement: in an infinite dimensional Hilbert space there always is an infinite dimensional subspace of points which are badly behaved with respect to projections on suitably chosen three subspaces.

Lemma 3.1. *There exists a sequence $k_1, k_2, \dots \in \{1,2,3\}$ with the following property. Suppose X is an infinite dimensional subspace of a Hilbert space H so that the Hilbert dimension of $X^\perp$ is at least as large as that of X. Then there exist three closed subspaces $X_1, X_2, X_3 \subset H$, so that for each $0 \neq z_0 \in X$ the sequence of iterates $\{z_n\}_{n=1}^\infty$ defined by $z_n = X_{k_n} z_{n-1}$ does not converge in norm.*

Proof. Let $k_1, k_2, \dots \in \{1,2,3\}$ be as in Theorem 2.6. Let $\{w^\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis of X. Let F_λ, $\lambda \in \Lambda$ be pairwise orthogonal closed infinite dimensional subspaces of $X^\perp$. Define $E_\lambda = w_\lambda \vee F_\lambda$. In each E_λ choose closed subspaces $X_1^\lambda, X_2^\lambda, X_3^\lambda \subset E_\lambda$ as in Theorem 2.6. For $j \in \{1,2,3\}$ define $X_j = \bigvee_{\lambda\in\Lambda} X_j^\lambda$.

Let $0 \neq z_0 = \sum_{\lambda\in\Lambda} t_\lambda w^\lambda \in X$ be given. Choose $\alpha \in \Lambda$ so that $t_\alpha \neq 0$ and write $z_0 = tw_0 + u_0$, where $t = t_\alpha$, and $w_0 = w^\alpha$. Since the spaces E_λ are pairwise orthogonal, the iterates of the point z_0 can be easily expressed using the iterates of w_0 and u_0, namely $z_n = tw_n + u_n$. Moreover,

$$\|z_n\|^2 = t^2\|w_n\|^2 + \|u_n\|^2 \geq t^2\|w_n\|^2,$$

since $w_n \in E_\alpha$ and $u_n \in E_\alpha^\perp$. Hence, the norm of the sequence $\{z_n\}$ stays bounded away from zero. As $\{z_n\}$ converges weakly to zero by [AA], it does not converge in norm. □

Next we will construct five subspaces of an infinite dimensional Hilbert space H, so that for each $0 \neq x \in H$ a certain sequence of iterates of projections of x on these five spaces does not converge in norm. We will use the following elementary observation.

Lemma 3.2. *Let H be a Hilbert space and let X be a closed subspace of H so that the Hilbert dimensions of X and $X^\perp$ are the same. Then there exists a closed subspace Y of H so that $Xz \neq 0$ or $XYz \neq 0$ for each $0 \neq z \in H$.*

Proof. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ and $\{f_\lambda\}_{\lambda\in\Lambda}$ be the orthonormal bases of X and $X^\perp$ respectively. Then $\{e_\lambda + f_\lambda\}_{\lambda\in\Lambda}$ is an orthogonal sequence; let Y be its closed linear span. Suppose $0 \neq z \in H$ is so that $Xz = 0$. Then

$z = \sum \langle f_\lambda, z \rangle f_\lambda$ with some $\langle f_\alpha, z \rangle \neq 0$. Hence

$$XYz = X\left(\sum_{\lambda\in\Lambda} \langle f_\lambda, z\rangle \frac{e_\lambda + f_\lambda}{2}\right) = \frac{1}{2}\sum_{\lambda\in\Lambda}\langle f_\lambda, z\rangle e_\lambda \neq 0.$$

□

Theorem 3.3. *Let H be an infinite dimensional Hilbert space. Then there exist five closed subspaces of H so that for every $0 \neq z \in H$ some sequence of iterates of z defined by the orthogonal projections on those subspaces does not converge in norm.*

More precisely, there exists a sequence $k_1, k_2, \cdots \in \{1, 2, 3\}$ with the following property. Every infinite dimensional Hilbert space H has closed subspaces X, Y, X_1, X_2, X_3 so that if $0 \neq z \in H$, and $u_0 = Xz$, $v_0 = XYz$, then at least one of the sequences of iterates $\{u_n\}_{n=1}^\infty$ or $\{v_n\}_{n=1}^\infty$ defined by $u_n = X_{k_n} u_{n-1}$, $v_n = X_{k_n} v_{n-1}$ does not converge in norm.

Proof. Let $k_1, k_2, \cdots \in \{1, 2, 3\}$ be as in Lemma 3.1. Choose a closed subspace X of H so that the Hilbert dimensions of X and $X^\perp$ are the same. Choose X_1, X_2, X_3 according to Lemma 3.1 and Y according to Lemma 3.2. □

A stronger result with only three subspaces can be obtained in a non-constructive way. First we will show that if the iterates of three projections of a certain point do not converge in norm, then there is a closed infinite dimensional subspace of such points. We recall a couple of elementary facts about projections we will need in the proof.

Let H be a Hilbert space. A continuous linear mapping $P : H \to H$ is called a projection if $P^2 = P$. It is an orthogonal projection if the range and the kernel of P are orthogonal. It is easy to see that a projection is orthogonal if and only if it is self adjoint.

Let $X, Z \subset H$ be closed subspaces so that $Xz \in Z$ for each $z \in Z$. Then $XZ = ZX$ is the orthogonal projection onto $X \cap Z$.

Conversely, assume that the orthogonal projections onto X and Z commute. Then again, XZ is the orthogonal projection onto $X \cap Z$.

Lemma 3.4. *Let H be a Hilbert space and let $X_1, X_2, X_3 \subset H$ be three of its closed subspaces. Suppose $w_0 \in H$ and $k_1, k_2, \cdots \in \{1, 2, 3\}$ are so that the sequence defined by $w_n = X_{k_n} w_{n-1}$ does not converge in norm. Then there exists a closed infinite dimensional subspace L of H and closed subspaces $Y_1, Y_2, Y_3 \subset L$ so that for every $0 \neq u_0 \in L$ there exist $j_1, j_2, \cdots \in \{1, 2, 3\}$ so that the sequence defined by $u_n = Y_{j_n} u_{n-1}$ does* not *converge in norm.*

Proof. Let Z be the set of all $z_0 \in H$ so that for every sequence $j_1, j_2, \dots \in \{1,2,3\}$ the sequence defined by $z_n = X_{j_n} z_{n-1}$ does converge in norm. Clearly, Z is a linear subspace; we observe that it is also closed. Indeed, assume that $\lim_{i\to\infty} y_0^i = y_0$ for some $y_0^i \in Z$ and that $j_1, j_2, \dots \in \{1,2,3\}$. Define recursively the sequences $y_n = X_{j_n} y_{n-1}$ and $y_n^m = X_{j_n} y_{n-1}^m$ for $m \in \mathbb{N}$. For each $\varepsilon > 0$ choose $m \in \mathbb{N}$ so that $\|y_0^m - y_0\| < \varepsilon$ and $N \in \mathbb{N}$ so that $\|y_k^m - y_l^m\| < \varepsilon$ for all $k, l > N$. Then $\|y_k - y_l\| \le \|y_k - y_k^m\| + \|y_k^m - y_l^m\| + \|y_l^m - y_l\| < 3\varepsilon$, since the composition of projections is 1-Lipschitz. Hence the sequence $\{y_n\}_{n=0}^\infty$ is Cauchy and therefore convergent, and $y_0 \in Z$.

From the definition of Z it follows that if $z \in Z$, then also $X_i z \in Z$ for $i \in \{1,2,3\}$. Thus $X_i Z = Z X_i$ is an orthogonal projection onto $X_i \cap Z$. Define $L = Z^\perp$. Then $L = Id_H - Z$ for the orthogonal projection onto L, hence L commutes with X_i for $i \in \{1,2,3\}$. Therefore $LX_i = X_i L$ is the orthogonal projection onto $Y_i = L \cap X_i$.

Assume $0 \neq u_0 \in L$. Since $L \cap Z = \{0\}$, there exist $j_1, j_2, \dots \in \{1,2,3\}$ so that the sequence defined by

$$u_n = X_{j_n} u_{n-1} = Y_{j_n} u_{n-1}$$

does not converge in norm, hence L is infinite dimensional by [Pr]. $\square$

We are now ready to prove the main theorem of our paper.

Theorem 3.5. *Every infinite dimensional Hilbert space H contains three closed subspaces X_1, X_2, X_3 with the following property. For every $0 \neq w_0 \in H$ there is a sequence $k_1, k_2, \dots \in \{1,2,3\}$ so that the sequence of iterates defined by $w_n = X_{k_n} w_{n-1}$ does not converge in norm.*

Proof. Assume first H is separable. According to Theorem 2.6, there exist $w_0 \in H$ and $k_1, k_2, \dots \in \{1,2,3\}$ so that the sequence defined by $w_n = X_{k_n} w_{n-1}$ does not converge in norm. Let L be the infinite dimensional subspace of H obtained in Lemma 3.4. Since H is separable, L and H are isometric, and the required subspaces of L, alias H, exist.

For a non-separable Hilbert space H we choose pairwise orthogonal, separable, infinite dimensional, closed spaces $H_\lambda \subset H$ so that

$$H = \bigvee_{\lambda\in\Lambda} H_\lambda.$$

In each H_λ we choose closed subspaces $X_1^\lambda, X_2^\lambda, X_3^\lambda$ according to the separable case of the theorem. We define

$$X_i = \bigvee_{\lambda\in\Lambda} X_i^\lambda, \ i \in \{1,2,3\}.$$

If $0 \neq w_0 \in H$, then $w_0 = \sum_{\lambda\in\Lambda} w_0^\lambda$ for some $w_\lambda \in H_\lambda$. Since $0 \neq w_0^\alpha$ for some $\alpha \in \Lambda$, there exist $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $w_n^\alpha = X_{k_n}^\alpha w_{n-1}^\alpha = X_{k_n} w_{n-1}^\alpha$ does not converge in norm. As

$$w_n = X_{k_n} w_{n-1} = \sum_{\lambda\in\Lambda} w_n^\lambda,$$

$\|w_n - w_m\| \geq \|w_n^\alpha - w_m^\alpha\|$ and the sequence $\{w_n\}_{n=0}^\infty$ does not converge either. □

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